

COM 251 Logic Design and Circuits

Combinational Logic Design: Canonical Forms

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Acknowledgement

The slides have been based in-part upon original slides of a number of books including:

- *Digital Design with RTL Design, Verilog and VHDL*, 2nd ed., Frank Vahid, John Wiley, 2011.
- *Logic and Computer Design Fundamentals*, 4th Ed., M. Morris Mano, C. Kime, Prentice Hall, 2008.

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Canonical Form – Sum of Minterms

- Truth tables too big for numerous inputs
- Use standard form of equation instead
 - Known as **canonical form**
 - Regular algebra: group terms of polynomial by power
 - $ax^2 + bx + c$ ($3x^2 + 4x + 2x^2 + 3 + 1 \rightarrow 5x^2 + 4x + 4$)
 - Boolean algebra: create sum of minterms
 - **Minterm**: product term with every function literal appearing exactly once, in true or complemented form
 - Just multiply-out equation until sum of product terms
 - Then expand each term until all terms are minterms

Q: Determine if $F(a,b)=ab+a'$ is equivalent to $F(a,b)=a'b'+a'b+ab$, by converting first equation to canonical form (second already is)

$F = ab+a'$ (already sum of products)

$F = ab + a'(b+b')$ (expanding term)

$F = ab + a'b + a'b'$ (Equivalent – same three terms as other equation)

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Canonical Form – Sum of Minterms

- Q: Determine whether the functions $G(a,b,c,d,e) = abcd + a'bcde$ and $H(a,b,c,d,e) = abcde + abcde' + a'bcde + a'bcde(a' + c)$ are equivalent.

$$G = abcd + a'bcde$$

$$G = abcd(e+e') + a'bcde$$

$$G = abcde + abcde' + a'bcde$$

$$G = a'bcde + abcde' + abcde \text{ (sum of minterms form)}$$

Equivalent

$$H = abcde + abcde' + a'bcde + a'bcde(a' + c)$$

$$H = abcde + abcde' + a'bcde + a'bcdea' + a'bcdec$$

$$H = abcde + abcde' + a'bcde + a'bcde + a'bcde$$

$$H = abcde + abcde' + a'bcde$$

$$H = a'bcde + abcde' + abcde$$

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Compact Sum of Minterms Representation

- List each minterm as a number
- Number determined from the binary representation of its variables' values
 - a'bcde corresponds to 01111, or 15
 - abcde' corresponds to 11110, or 30
 - abcde corresponds to 11111, or 31
- Thus, $H = a'bcde + abcde' + abcde$ can be written as:
 - $H = \sum m(15,30,31)$
 - "H is the sum of minterms 15, 30, and 31"

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More on Canonical Forms

- What are Canonical Forms?
- Minterms and Maxterms
- Index Representation of Minterms and Maxterms
- Sum-of-Minterm (SOM) Representations
- Product-of-Maxterm (POM) Representations
- Representation of Complements of Functions
- Conversions between Representations

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Canonical Forms

- It is useful to specify Boolean functions in a form that:
 - Allows comparison for equality.
 - Has a correspondence to the truth tables
- Canonical Forms in common usage:
 - *Sum of Minterms (SOM)*
 - *Product of Maxterms (POM)*

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Minterms

- Minterms are AND terms with every variable present in either *true* or *complemented* form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., \bar{x}), there are 2^n minterms for n variables.
- Example: 2 variables (X and Y) produce $2 \times 2 = 4$ combinations:
 - XY (both normal)
 - $X\bar{Y}$ (X normal, Y complemented)
 - $\bar{X}Y$ (X complemented, Y normal)
 - $\bar{X}\bar{Y}$ (both complemented)
- Thus there are 4 minterms of 2 variables.

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Maxterms

- Maxterms are OR terms with every variable in true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., \bar{x}), there are 2^n maxterms for n variables.
- Example: 2 variables (X and Y) produce $2 \times 2 = 4$ combinations:

$\mathbf{X + Y}$ (both normal)

$\mathbf{X + \bar{Y}}$ (x normal, y complemented)

$\mathbf{\bar{X} + Y}$ (x complemented, y normal)

$\mathbf{\bar{X} + \bar{Y}}$ (both complemented)

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Maxterms and Minterms

- Examples: 2 variable minterms and maxterms.

Index	Minterm	Maxterm
0	$\bar{x} \bar{y}$	$x + y$
1	$\bar{x} y$	$x + \bar{y}$
2	$x \bar{y}$	$\bar{x} + y$
3	$x y$	$\bar{x} + \bar{y}$

- The index above is important for describing which variables in the terms are true and which are complemented.

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Standard Order

- Minterms and maxterms are designated with a *subscript*
- The subscript is a number, corresponding to a binary pattern.
- The bits in the pattern represent the complemented or normal state of each variable listed in a standard order.
- All variables will be present in a minterm or maxterm and will be listed in the **same order (usually alphabetically)**
- Example: For variables a, b, c :
 - Maxterms: $(a + b + \bar{c})$, $(a + b + c)$
 - Terms: $(b + a + c)$, $a \bar{c} b$, and $(c + b + a)$ are NOT in standard order.
 - Minterms: $a \bar{b} c$, $a b c$, $\bar{a} \bar{b} c$
 - Terms: $(a + c)$, $\bar{b} c$, and $(\bar{a} + b)$ do not contain all variables

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Purpose of the Index

- The index for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true form or complemented form.
- For Minterms:
 - “1” means the variable is “Not Complemented” and
 - “0” means the variable is “Complemented”.
- For Maxterms:
 - “0” means the variable is “Not Complemented” and
 - “1” means the variable is “Complemented”.

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Index Example in 3 Variables

- Variables: X, Y, and Z.
- The standard order is X, then Y, then Z.
- The Index 0 (base 10) = 000 (base 2) for 3 variables.
- All 3 variables are complemented for minterm 0 ($\overline{X}, \overline{Y}, \overline{Z}$) and no variables are complemented for Maxterm 0 (X,Y,Z).
 - Minterm 0, called m_0 is $\overline{X}\overline{Y}\overline{Z}$.
 - Maxterm 0, called M_0 is $(X + Y + Z)$.
 - Minterm 6 ?
 - Maxterm 6 ?

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Index Examples – Four Variables

Index Binary Minterm Maxterm

i	Pattern	m_i	M_i
0	0000	$\overline{a}\overline{b}\overline{c}\overline{d}$	$a + b + c + d$
1	0001	$\overline{a}\overline{b}\overline{c}d$?
3	0011	?	$a + b + \overline{c} + \overline{d}$
5	0101	$\overline{a}b\overline{c}d$	$a + \overline{b} + c + \overline{d}$
7	0111	?	$a + \overline{b} + \overline{c} + \overline{d}$
10	1010	$a\overline{b}\overline{c}d$	$\overline{a} + b + \overline{c} + d$
13	1101	$ab\overline{c}d$?
15	1111	$abcd$	$\overline{a} + \overline{b} + \overline{c} + \overline{d}$

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Minterm and Maxterm Relationship

- Review: DeMorgan's Theorem**
 $\overline{x \cdot y} = \overline{x} + \overline{y}$ and $\overline{x + y} = \overline{x} \cdot \overline{y}$
- Two-variable example:**
 $M_2 = \overline{x} + y$ and $m_2 = x \cdot \overline{y}$
 Thus M_2 is the complement of m_2 and vice-versa.
- Since DeMorgan's Theorem holds for n variables, the above holds for terms of n variables
- giving:

$$M_i = \overline{m_i} \text{ and } m_i = \overline{M_i}$$
 Thus M_i is the complement of m_i .

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Function Tables for Both

Minterms of 2 variables

x y	m_0	m_1	m_2	m_3
00	1	0	0	0
01	0	1	0	0
10	0	0	1	0
11	0	0	0	1

Maxterms of 2 variables

x y	M_0	M_1	M_2	M_3
00	0	1	1	1
01	1	0	1	1
10	1	1	0	1
11	1	1	1	0

- Each column in the maxterm function table is the complement of the column in the minterm function table since M_i is the complement of m_i .

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Observations

- In the function tables:
 - Each minterm has one and only one 1 present in the 2^n terms (a minimum of 1s). All other entries are 0.
 - Each maxterm has one and only one 0 present in the 2^n terms. All other entries are 1 (a maximum of 1s).
- We can implement any function by "ORing" the minterms corresponding to "1" entries in the function table. These are called the minterms of the function.
- We can implement any function by "ANDing" the maxterms corresponding to "0" entries in the function table. These are called the maxterms of the function.
- This gives us 2 canonical forms:
 - Sum of Minterms (SOM)
 - Product of Maxterms (POM)
 for stating any Boolean function.

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Minterm Function Example

- Example: Find $F_1 = m_1 + m_4 + m_7$**
- $F_1 = \bar{x} \bar{y} z + x \bar{y} \bar{z} + x y z$**

x y z	index	$m_1 + m_4 + m_7 = F_1$
0 0 0	0	0 + 0 + 0 = 0
0 0 1	1	1 + 0 + 0 = 1
0 1 0	2	0 + 0 + 0 = 0
0 1 1	3	0 + 0 + 0 = 0
1 0 0	4	0 + 1 + 0 = 1
1 0 1	5	0 + 0 + 0 = 0
1 1 0	6	0 + 0 + 0 = 0
1 1 1	7	0 + 0 + 1 = 1

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Minterm Function Example

- $F(A, B, C, D, E) = m_2 + m_9 + m_{17} + m_{23}$**
- $F(A, B, C, D, E) =$**

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Maxterm Function Example

- Example: Implement F1 in maxterms:**
- $F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$**
- $F_1 = (x + y + z) \cdot (x + \bar{y} + z) \cdot (x + \bar{y} + \bar{z})$**
 $\cdot (\bar{x} + y + \bar{z}) \cdot (\bar{x} + \bar{y} + z)$

x y z	i	$M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6 = F_1$
0 0 0	0	0 · 1 · 1 · 1 · 1 = 0
0 0 1	1	1 · 1 · 1 · 1 · 1 = 1
0 1 0	2	1 · 0 · 1 · 1 · 1 = 0
0 1 1	3	1 · 1 · 0 · 1 · 1 = 0
1 0 0	4	1 · 1 · 1 · 1 · 1 = 1
1 0 1	5	1 · 1 · 1 · 0 · 1 = 0
1 1 0	6	1 · 1 · 1 · 1 · 0 = 0
1 1 1	7	1 · 1 · 1 · 1 · 1 = 1

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Maxterm Function Example

- $F(A,B,C,D) = M_3 \cdot M_8 \cdot M_{11} \cdot M_{14}$
- $F(A, B,C,D) =$

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Canonical Sum of Minterms

- Any Boolean function can be expressed as a Sum of Minterms.
 - For the function table, the minterms used are the terms corresponding to the 1's
 - For expressions, expand all terms first to explicitly list all minterms. Do this by “ANDing” any term missing a variable v with a term $(v + \bar{v})$.
- Example: Implement $f = x + \bar{x} \bar{y}$ as a sum of minterms.
 - First expand terms: $f = x(y + \bar{y}) + \bar{x} \bar{y}$
 - Then distribute terms: $f = xy + x\bar{y} + \bar{x} \bar{y}$
 - Express as sum of minterms: $f = m_3 + m_2 + m_0$

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Another SOM Example

- Example: $F = A + \bar{B} C$
- There are three variables, A, B, and C which we take to be the standard order.
- Expanding the terms with missing variables:
 - Collect terms (removing all but one of duplicate terms):
 - Express as SOM:

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Shorthand SOM Form

- From the previous example, we started with:
$$F = A + \bar{B} C$$
- We ended up with:
$$F = m_1 + m_4 + m_5 + m_6 + m_7$$
- This can be denoted in the formal shorthand:
$$F(A, B, C) = \Sigma_m(1,4,5,6,7)$$
- Note that we explicitly show the standard variables in order and drop the “m” designators.

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Canonical Product of Maxterms

- Any Boolean Function can be expressed as a Product of Maxterms (POM).
 - For the function table, the maxterms used are the terms corresponding to the 0's.
 - For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law, "ORing" terms missing variable v with a term equal to $V \cdot \bar{V}$ and then applying the distributive law again.
- Example: Convert to product of maxterms:

$$f(x, y, z) = x + \bar{x} \bar{y}$$
 Apply the distributive law:

$$x + \bar{x} \bar{y} = (x + \bar{x})(x + \bar{y}) = 1 \cdot (x + \bar{y}) = x + \bar{y}$$
 Add missing variable z :

$$x + \bar{y} + z \cdot \bar{z} = (x + \bar{y} + z)(x + \bar{y} + \bar{z})$$
 Express as POM: $f = M_2 \cdot M_3$

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Another POM Example

- Convert to Product of Maxterms:

$$f(A, B, C) = A \bar{C} + BC + \bar{A} \bar{B}$$
- Use $x + y z = (x+y) \cdot (x+z)$ with $x = (A \bar{C} + BC)$, $y = \bar{A}$, and $z = \bar{B}$ to get:

$$f = (A \bar{C} + BC + \bar{A})(A \bar{C} + BC + \bar{B})$$
- Then use $x + \bar{x} y = x + y$ to get:

$$f = (\bar{C} + BC + \bar{A})(A \bar{C} + C + \bar{B})$$
 and a second time to get:

$$f = (\bar{C} + B + \bar{A})(A + C + \bar{B})$$
- Rearrange to standard order,

$$f = (\bar{A} + B + \bar{C})(A + \bar{B} + C)$$
 to give $f = M_5 \cdot M_2$

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Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- Example: Given $F(x, y, z) = \Sigma_m(1,3,5,7)$

$$\bar{F}(x, y, z) = \Sigma_m(0,2,4,6)$$

$$\bar{F}(x, y, z) = \Pi_M(1,3,5,7)$$

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Conversion Between Forms

- To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:
 - Find the function complement by swapping terms in the list with terms not in the list.
 - Change from products to sums, or vice versa.
- Example: Given F as before: $F(x, y, z) = \Sigma_m(1,3,5,7)$
- Form the Complement: $\bar{F}(x, y, z) = \Sigma_m(0,2,4,6)$
- Then use the other form with the same indices – this forms the complement again, giving the other form of the original function: $F(x, y, z) = \Pi_M(0,2,4,6)$

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Standard Forms

- **Standard Sum-of-Products (SOP) form:** equations are written as an OR of AND terms
- **Standard Product-of-Sums (POS) form:** equations are written as an AND of OR terms
- **Examples:**
 - SOP: $A B C + \bar{A} \bar{B} C + B$
 - POS: $(A + B) \cdot (A + \bar{B} + \bar{C}) \cdot C$
- These “mixed” forms are neither SOP nor POS
 - $(A B + C) (A + C)$
 - $A B \bar{C} + A C (A + B)$

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Standard Sum-of-Products (SOP)

- A sum of minterms form for n variables can be written down directly from a truth table.
 - Implementation of this form is a two-level network of gates such that:
 - The first level consists of n -input AND gates, and
 - The second level is a single OR gate (with fewer than 2^n inputs).
- This form often can be simplified so that the corresponding circuit is simpler.

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Standard Sum-of-Products (SOP)

- A Simplification Example:
- $F(A, B, C) = \Sigma m(1,4,5,6,7)$
- Writing the minterm expression:

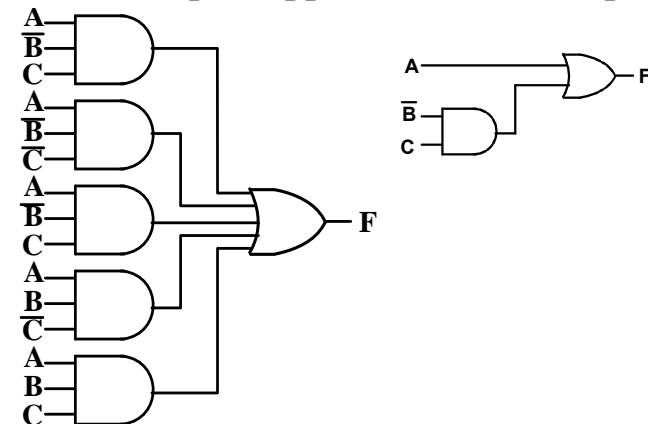
$$F = \bar{A} \bar{B} C + A \bar{B} \bar{C} + A \bar{B} C + A B \bar{C} + A B C$$
- Simplifying:

$$F =$$
- Simplified F contains 3 literals compared to 15 in minterm F

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AND/OR Two-level Implementation of SOP Expression

- The two implementations for F are shown below – it is quite apparent which is simpler!



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SOP and POS Observations

- The previous examples show that:
 - Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity
 - Boolean algebra can be used to manipulate equations into simpler forms.
 - Simpler equations lead to simpler two-level implementations
- Questions:
 - How can we attain a “simplest” expression?
 - Is there only one minimum cost circuit?
 - The next part will deal with these issues.